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Invariance properties of the q-oscillator algebra: q-analogue of von Neumann's theorem

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Abstract. Transformations which leave the q-oscillator algebra invariant are considered. In general there exists no uniqueness theorem for the q-oscillator case. It is proven, however, that the analogue of von Neumann's theorem exists for a certain class of transformations. An essential part of these transformations is defined by the relation between the elements a^+a and N involving a constant. The latter can be interpreted as a central element of the q-oscillator algebra. The explicit form of non-unitary transformations with invariance of the algebra is obtained as well.

1. Introduction

Quantum groups [1–3] and q-oscillators [4–10] have recently been the subject of intensive study. The simplest q-oscillator satisfies the relations:

$$aa^+ - qa^+a = q^{-N} \qquad q \in \mathrm{R} \text{ or } q \in C,\ |q| = 1 \tag{1}$$

$$[N, a] = -a \qquad [N, a^+] = a^+ \qquad (a^+)^+ = a \qquad N^+ = N\,. \tag{2}$$

In this paper we consider the transformations, which leave the relations (1), (2) (for brevity we call them 'q-algebra') unchanged. We restrict ourselves to the case of irreducible representations only. It is well known, that in the case of Heisenberg algebra

$$bb^+ - b^+b = 1 \qquad [N, b] = -b \qquad (b^+)^+ = b \qquad N = b^+b \tag{3}$$

the analogous problem in Hilbert space is solved by the von Neumann theorem (see, e.g. [11]). We prove, that in the case of q-algebra representations in some inner product space all the transformations, which leave this algebra invariant are reduced to a combination of transformations described by a unitary operator (the analogue of von Neumann's theorem) and of transformations, the explicit form of which are known (cf (30)). In addition, we obtain formulae relating the $a(q)$ with different values of q.

In section 2 a general formula, which relates aa^+ and a^+a with N is obtained (cf (21)). Here the space, in which the operators a and a^+ act, is realized as the space constructed out of eigenvectors of operator N. (As usual, one assumes that the operator N has at least one eigenvalue). In the general case the relation between $aa^+(a^+a)$ and N contains some real constant C. This constant is related to the value of the central element found in [9, 10]

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for the irreducible representation we are working in. Due to this fact, the transformations which leave the q-algebra invariant, change in general both C and N but can also leave one of the two unchanged.

The unitary irreducible representations have been classified in [9, 10]. However, we obtain new ones with indefinite metric. In the general case, i.e. for arbitrary C, the space in which the operators a and a^+ act, is a space with an indefinite metric [12], and unitarity is understood as the one with respect to indefinite scalar product. We recall, that spaces with indefinite metric could be of interest in quantum gauge field representation theory where they have been widely used (cf e.g. [13, 14]).

In section 3 the analogue of von Neumann's theorem is proved. The proven theorem is actually applicable to any arbitrary algebra, in which the relations (2) are satisfied and when there exists one-to-one relation between the operators aa^+, a^+a and N. The concrete form of commutation relations does not play any role, i.e. the relation (1) can be replaced by a more general one like (27).

In section 4 we obtain the explicit form of transformations which relate the operators a and a^+ for different values of C, but for the same N. It is shown that the transformation: $a \to \tilde{a}$, $C \to \tilde{C}$, $N \to N$ is possible only when the constants C and $\tilde{C}$ are related to each other in a specific way. Namely, the whole region of C values is divided into a series of intervals, such that the operators belonging to one of these intervals can be related to each other. If the value of C and $\tilde{C}$ belong to different intervals, then the corresponding operators constitute the realization of the q-algebra in different spaces. In the same section we obtain formulae relating $a(q)$ with different q and we find the regions of q values when such a transformation is possible.

2. General relations between the operators aa^+, a^+a and N

To obtain a general formula which relates $aa^+(a^+a)$ with N, we notice that

$$[N+1] - q[N] = q^{-N} \qquad [x] = \frac{q^x - q^{-x}}{q - q^{-1}} . \tag{4}$$

We restrict ourselves to irreducible representations. Thus aa^+ and a^+a can be written in the form

$$aa^+ = [N+1] + q\varphi(a, a^+) \qquad a^+a = [N] + \varphi(a, a^+) . \tag{5}$$

As we consider an irreducible representation of q-algebra, N is some function of a and a^+. We prove, that actually $\varphi(a, a^+)$ depends only on N,

$$\varphi(a, a^+) = \varphi(N) \tag{6}$$

and then we obtain the explicit form of $\varphi(N)$. At first let us show that

$$\varphi(a, a^+) = \varphi(aa^+, N) . \tag{7}$$

Since $[aa^+, N] = [a^+a, N] = 0$ according to (2), one has $[\varphi(a, a^+), N] = 0$. Equation (7) is a special case of the statement that any function $\psi(a, a^+)$, which satisfies the relation

$$[\psi(a, a^+), N] = 0 \tag{8}$$

is actually a function of only aa^+ and N,

$$\psi(a, a^+) = \psi(aa^+, N) . \tag{9}$$

To prove (9), we notice that the expansion of an arbitrary (non-singular) function $\psi(a, a^+)$ into a power series with respect to a and a^+ has the form

$$\psi(a, a^+) = \sum_{n,k,\nu} C^{\nu}_{n,k} P_{\nu}(a^n, a^{+k}) . \tag{10}$$

Index ν designates the ordering of operators a and a^+ in the polynomial. It is easy to see, that the commutation relations (2) allow us to rewrite (10) as

$$\psi(a, a^+) = \sum_n \Phi^{(1)}_n (N) a^n + \sum_n \Phi^{(2)}_n (N) a^{+n} + \Phi(aa^+, N) . \tag{11}$$

Taking into account that, according to (2), $[N, a^n] = -na^n$ and $[N, a^{+n}] = na^{+n}$, we see that the condition (8) is fulfilled only if all the $\Phi^{(i)}_n$, $i = 1, 2$ vanish, i.e. if the function $\psi(a, a^+)$ is of the form (9). Thus (7) is proven. In order to pass from (7) to (6), we notice that for the special case $\psi(a, a^+) = N$, (9) becomes

$$N = \mu(aa^+, N) . \tag{12}$$

Equation (12) corresponds to a functional dependence as it is implied by (6). To obtain the form of the function $\psi(N)$, we prove that it satisfies the relation

$$\varphi(N + 1) = q\varphi(N) . \tag{13}$$

Indeed, according to (5), one has

$$aa^+a = [N + 1]a + q\varphi(N)a \qquad aa^+a = a[N] + a\varphi(N) = [N + 1]a + \varphi(N + 1)a .$$

In the latter equation we have used the fact that $a\psi(N) = \psi(N + 1)a$, which follows from (2) for an arbitrary function $\psi(N)$. Comparison of the two expressions for aa^+a leads to (13).

To solve (13), we first find the spectrum of operator N. We notice that if $|\alpha\rangle$ is an eigenvector of the operator $N : N|\alpha\rangle = \alpha|\alpha\rangle$, then by making the transformation $a \to q^{\alpha/2}a$, $a^+ \to q^{\alpha/2}a^+$ and $N \to N' = N - \alpha$, which leaves the q-algebra invariant, we obtain that

$$N'|\alpha\rangle = 0 . \tag{14}$$

The latter transformation is supposed to give sense to q taken to any power. In the other cases we can prove, that spectrum of N is $\alpha + Z$, $0 \leqslant \alpha < 1$. In this case the results remain unchanged up to evident modification. Thus, by performing the above transformation we can arrive at the case when the operator N has the eigenvector $|0\rangle$, such that

$$N|0\rangle = 0 \qquad \langle 0|0\rangle = \pm 1 . \tag{15}$$

In the inner product space the case $\langle 0|0\rangle = 0$ is also possible, but then (cf (23)) all scalar products are equal to zero (isotropic space).

Using (1), (2) and (15) one sees that there exists a set of vectors $|n\rangle$, such that

$$N|n\rangle = n|n\rangle \qquad n = 0, \pm 1, \pm 2, \ldots, \pm\infty \tag{16}$$

where the vectors $|n\rangle$ and $|n-1\rangle$ are related as

$$|n\rangle = C_n a^+ |n-1\rangle \qquad |n-1\rangle = \tilde{C}_n a|n\rangle\,. \tag{17}$$

The normalization constants C_n and $\tilde{C}_n$ are chosen so that

$$\langle n|m\rangle = C_{nm}\delta_{nm} \qquad C_{nn} = \pm 1\,. \tag{18}$$

If $\langle n|n\rangle = 0$ for some n, then it can be shown that vectors $P(a, a^+)|n\rangle$ form invariant subspace in contradiction with irreducibility of the representation under consideration, $P(a, a^+)$ being an arbitrary polynomial. Notice that due to hermiticity of the operator N (in the general case with respect to indefinite scalar product), $\langle n|m\rangle = 0$ for $n \neq m$.

Thus as a result of the operator N being a self-conjugate one, the vectors $|n\rangle$ constitute an orthonormal basis.

We emphasize that (16) follows only from (15) and the commutation relations (2). The constant values C_n and $\tilde{C}_n$ are determined from (18). Thus, the space in which the operators a and a^+ act is the space of all finite linear combinations of vectors $|n\rangle$ and their limiting values. In other words, it is the closure of the space of all finite linear combinations of the vectors $|n\rangle$. An important property of this space is its non-degeneracy, i.e. absence of vectors orthogonal to all the vectors of the space. Indeed, any vector of this space has the form: $|\alpha\rangle = \sum_n \alpha_n |n\rangle$. Clearly, among α_n there must exist some α_k, such that $\alpha_k \neq 0$. Then $\langle\alpha|k\rangle = \alpha_k \neq 0$. From the non-degeneracy of the space, the following statement, which plays a crucial role for derivation of the main results, follows: if A and B are two operators, such that

$$\langle\alpha|A|\beta\rangle = \langle\alpha|B|\beta\rangle \tag{19}$$

where $|\alpha\rangle$ and $|\beta\rangle$ are arbitrary vectors, then $A = B$. Indeed, suppose that $A - B \neq 0$. Then there should exist a vector $|\beta\rangle$, such that $(A - B)|\beta\rangle = |\beta'\rangle \neq 0$. According to (19) then $|\beta'\rangle$ would be orthogonal to all the vectors of the space and because of non-degeneracy it should vanish.

Consider now the solution of (13). Clearly

$$\varphi(N)|k\rangle = q^k \varphi(N-k)|k\rangle = q^N \varphi(0)|k\rangle \tag{20}$$

where $\varphi(0) = C'$ is some number. Then

$$\varphi(N) = C' q^N\,. \tag{21}$$

Thus it is proven that

$$aa^+ = [N+1] + C'q^{N+1} \equiv [N+1]_c \qquad a^+a = [N]_c \tag{22}$$

where $[\mathrm{M}]_c = (Cq^{\mathrm{M}} - q^{-\mathrm{M}})(q - q^{-1})^{-1}$, $C' = (C-1)(q-q^{-1})^{-1}$. The constant C' is related to the central element of the q-oscillator algebra [9]. From the hermiticity of the operators aa^+ and N it follows that C' is a real number when q is real. If q is complex, it is easy to see, that $C' = 0$.

Equations (22) and (1) allow us to determine the constants C_n and $\tilde{C}_n$. A simple calculation shows that if $|\alpha_n^+\rangle = a^{+n}|0\rangle$, $|\alpha_n\rangle = a^n|0\rangle$, then

$$\langle\alpha_n^+|\alpha_n^+\rangle = [n]_c[n-1]_c \ldots [1]_c\langle 0|0\rangle \qquad \langle\alpha_n|\alpha_n\rangle = [-n+1]_c[-n+2]_c \ldots [0]_c\langle 0|0\rangle\,. \tag{23}$$

It is easy to see that $[n]_c$, $[-n]_c$ can have an arbitrary sign and therefore the space in general, i.e. for arbitrary q and C, is a space with indefinite metric (the properties of such spaces are described in detail in [12]).

3. von Neumann's theorem for the q-oscillator algebra

We now obtain the main result, namely we prove the following theorem: let a_i and a_i^+, $i = 1, 2$, be two irreducible representations of the q-oscillator algebra (1) and (2) in spaces J_i correspondingly. Let the relations between $a_i a_i^+$, $a_i^+ a_i$ and N be given by (22) in which C does not depend on i. If in each of the spaces J_i there exists a vector $|0_i\rangle$, such that $N_i|0_i\rangle = 0$ then there exists a unitary (or quasi-unitary) operator V, such that

$$a_2 = V a_1 \mathrm{V}^{-1} \qquad a_2^+ = V a_1^+ V^{-1} \qquad N_2 = V N_1 V^{-1}\,. \tag{24}$$

By definition, quasi-unitary operator is an operator satisfying the relations: $VV^+ = V^+V = -1$. We emphasize that unitarity of operator V means the unitarity with respect to indefinite scalar product. It is sufficient to consider only the case when $\langle 0_1|0_1\rangle = \langle 0_2|0_2\rangle$, because the proof in the case $\langle 0_1|0_1\rangle = -\langle 0_2|0_2\rangle$ is similar. If $\langle 0_1|0_1\rangle = \langle 0_2|0_2\rangle$, then V is an unitary operator; if $\langle 0_1|0_1\rangle = -\langle 0_2|0_2\rangle$, then V is a quasi-unitary one.

For the proof, we notice that to any arbitrary vector $|\alpha_1\rangle$ of the space J_1, one can put into correspondence a vector $|\alpha_2\rangle$ of the space J_2, which is expressed in terms of the vectors $|n_2\rangle$ in the same way as $|\alpha_1\rangle$ is expressed in terms of $|n_1\rangle$. Let us prove, that the above-mentioned correspondence preserves the scalar product, i.e. for arbitrary vectors we have

$$\langle\alpha_1|\beta_1\rangle = \langle\alpha_2|\beta_2\rangle\,. \tag{25}$$

To prove (25), we observe that for an arbitrary function $\psi(\mathrm{N})$ the following equation is valid:

$$\langle 0_1|\psi(\mathrm{N}_1)|0_1\rangle = \langle 0_2|\psi(\mathrm{N}_2)|0_2\rangle\,.$$

Using the commutation relations (2) and (22) we obtain that $\langle\alpha_i|\beta_i\rangle = \langle 0_i|\psi(\mathrm{N}_i)|0_i\rangle$, where the functional form of $\psi(\mathrm{N}_i)$ does not depend on i, i.e. (25) is proven.

Now introduce the operator V: $|\alpha_2\rangle = \mathrm{V}|\alpha_1\rangle$. According to (25), $\langle\alpha_1|V^+\mathrm{V}|\beta_1\rangle = \langle\alpha_1|\beta_1\rangle$. We prove that $V^+V = 1$, i.e. V is an isometric operator. Since the operator V provides with a one-to-one correspondence between the spaces J_1 and J_2, then there exists an inverse operator V^{-1}. In such a case, as easily seen, from the equality $V^+V = 1$ follows that $VV^+ = 1$, i.e. V is a unitary operator. The required (24) is then a direct consequence of unitarity of V and of (25). Indeed, let $|\beta_i\rangle = a_i|\beta_i'\rangle$. According to previously proven: $\langle\alpha_1|\beta_1\rangle = \langle\alpha_2|\beta_2\rangle$, but $|\alpha_2\rangle = V|\alpha_1\rangle$, $|\beta_2'\rangle = V|\beta_1'\rangle$, and consequently

$$\langle\alpha_1|a_1|\beta_1'\rangle = \langle\alpha_1|V^+ a_2 V|\beta_1'\rangle\,. \tag{26}$$

Taking into account (19), we see that (26) is equivalent to (24). The theorem is proven.

Let us mention two interesting points, that the specific form of the commutation relations (1) has not been used for the proof of the theorem. This means that the theorem remains valid if the q-algebra is replaced by a generalized q-algebra, in which (1) is replaced by

$$aa^+ - qa^+a = \xi(q, N) \tag{27}$$

(the relations (2) remaining, of course, the same). The only condition that the function $\xi(q, N)$ ought to satisfy, is the possibility of obtaining formulae analogous to (22).

Since for complex q values $C = 0$, then the Theorem is valid also in this case, including the exceptional values of $q = (-1)^{1/n}$.

Furthermore, if $C = 0$, then aa^+ and a^+a do not change with the replacement $q \to q^{-1}$. Thus (for $C = 0$) the theorem can be applied also for the transformation $a(q) \to a(q^{-1}), a^+(q) \to a^+(q^{-1})$, $N(q) \to N(q^{-1})$, to obtain that in this case $a(q)$ and $a(q^{-1})$, e.g. are related by a unitary transformation.

4. Non-unitary transformations which leave the q-algebra invariant

An arbitrary transformation in a given space which leaves the q-algebra invariant can also be chosen specifically so that it leaves either C or N unchanged.

Consider a transformation which does not change N. First of all we notice that if $\tilde{a}$, $\tilde{a}^+$, a and a^+ are operators which satisfy the relations (1) and (2) with the same N, then

$$\tilde{a} = \varphi(N)a \qquad \tilde{a}^+ = a^+\varphi(N)\,. \tag{28}$$

Assuming that the dependence of $\tilde{a}$ on a has a more general form, $\tilde{a} = \varphi(a, a^+)$, and repeating the arguments which led us to prove (6), we obtain that $\varphi(a, a^+) = \varphi(N)a$. Indeed, the general form of the function $\varphi(a, a^+)$ is determined by (11). It is easy to see that relation $[\tilde{a}, N] = -\tilde{a}$ is possible only in the case when all the terms in (11) vanish except $\Phi_1^{(1)}a$, since $[N, a^n] = -na^n$, $[N, a^{+n}] = na^{+n}$. Equations (28) and (20) allow us to determine the form of the operator $\varphi(N)$. Indeed,

$$\tilde{a}\tilde{a}^+ = \varphi(N)aa^+\varphi(N) = \varphi^2(N)aa^+ \tag{29}$$

and using $aa^+ = [N+1]_c$, $\tilde{a}\tilde{a}^+ = [N+1]_{\tilde{c}}$ (cf (22)), we obtain that

$$\varphi(N) = [N+1]_{\tilde{c}}^{1/2}[N+1]_c^{-1/2}\,. \tag{30}$$

We emphasize that the transformation (30) is meaningful only for such values of C and $\tilde{C}$, for which φ(N) is a well-defined operator. Consider this equation in more detail. Obviously, it is sufficient to study only the case $\langle 0|0\rangle = 1$.

(1) Let $1 < q < \infty$.

Then the sign of $[n]_c$ coincides with the sign of $Cq^n - q^{-n}$. If $C < 0$, then $[n]_c < 0$. If $C > 0$, then

$$[k]_c = 0 \qquad \text{if } C \equiv C_k = q^{-2k}\,. \tag{31}$$

From (31) one sees that $[n]_c > 0$, if $n > k$, and $[n]_c < 0$, if $n < k$. In this way the set of values of C is divided into the intervals R_k:

$$C \in \mathrm{R}_k \qquad \text{if } C_{k+1} < C < C_k\,. \tag{32}$$

(2) Let $0 < q < 1$.

Then the sign of $[n]_c$ is opposite to the sign of $Cq^n - q^{-n}$. If $C < 0$, then $[n]_c > 0$, i.e. our space is the usual Hilbert space. If $C > 0$, then the solution of $[n]_c = 0$ is, as before, given by (31). The set of values of C is divided into the intervals R'_k:

$$C \in \mathrm{R}'_k \qquad \text{if } C_k < C < C_{k+1}\,. \tag{33}$$

So if $C \neq C_k$ and $C > 0$, then the eigenvectors of operator N are $|n\rangle$, where $n = 0, \pm 1, \ldots, \pm\infty$. In this case if $C \in R_k$ (or $C \in R'_k$), then all the vectors $|n\rangle$ with $n > k$ are positive, and for $n < k$ $\langle n|n\rangle = (-1)^{k-n}$. When $C < 0$, then for $1 < q < \infty$ we have $\langle n|n\rangle = 1$ if $n = 2\ell$, and $\langle n|n\rangle = -1$ if $n = 2\ell + 1$. When $0 < q < 1$ and $C < 0$, then $\langle n|n\rangle = 1$. When $C = C_k$, then the chain of eigenvectors of the operator N is truncated at some vector $|n\rangle$ from below or from above depending on which of the two conditions

$a|n\rangle = 0$ or $a^+|n\rangle = 0$ takes place. At this point we use irreducibility of representation under consideration. In this case, by making the transformation $N \to N - n$, $a(q) \to q^{n/2}a(q)$ and $a^+(q) \to q^{n/2}a^+(q)$ we arrive at the case when the operator N has either the spectrum $(0, 1, \ldots, n, \ldots \infty)$ or $(-\infty, \ldots -n, \ldots -1, 0)$. The transformation (30) is well defined, if $\varphi^2(N)$ is a positive operator, i.e. $[n+1]_{\tilde{c}}[n+1]_c^{-1} > 0$ for any n. This obviously happens in the case when $\tilde{C} \in R_k$ and $C \in R_k$ (or $\tilde{C} \in R'_k$ and $C \in R'_k$). Thus (30) relates the operators $\tilde{a}$ and a, if both $\tilde{C}$ and C belong to the same interval R_k (or R'_k). In the opposite case operators a and $\tilde{a}$ act in different spaces.

In full analogy one can consider the question about the relation between the operators $a(q_1)$ and $a(q_2)$ with C and N unchanged. It is easy to see that the form of the formula (30) remains the same, but now

$$\varphi(N) = [N+1]_{q_1,c}^{1/2}\,[N+1]_{q_2,c}^{-1/2}\,. \tag{34}$$

Analogously to the forthmentioned, one can analyse how the set of q values gets divided into regions in which (34) is valid. In the special case, when $C = 1$, $q_1 = q$ and $q_2 = 1$, the formula (34) coincides with the one obtained in [8]. Finally, an equation analogous to (34) is valid also for the case with the transformation $q_1 \to q_2$, $C_1 \to C_2$ and $N \to N$.

To summarize, we have fully classified the transformations which leave the q-oscillator algebra invariant.

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